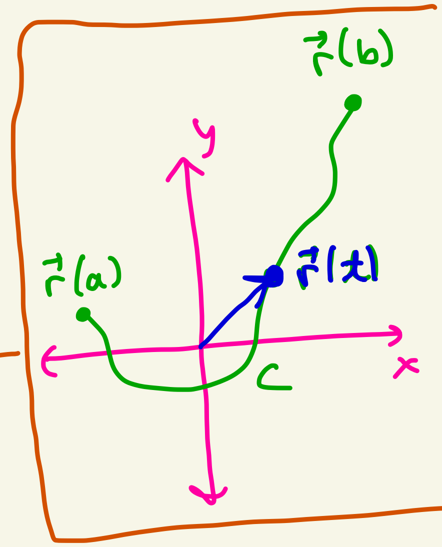
Topic 8 -
Line integrals

Part 1: Curves in 2d and 3d

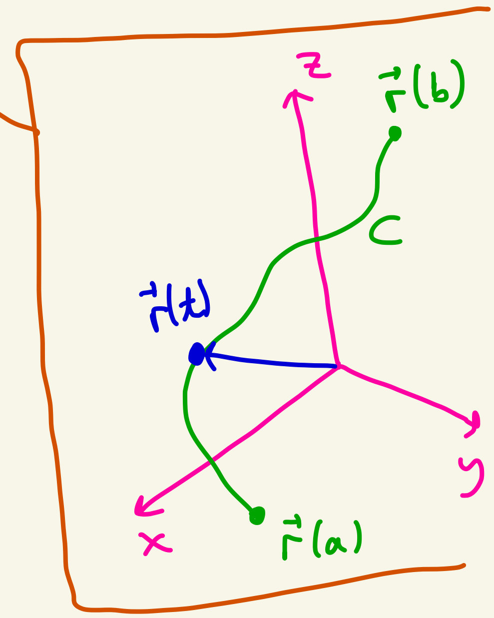


Recall that a vector function

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

or

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

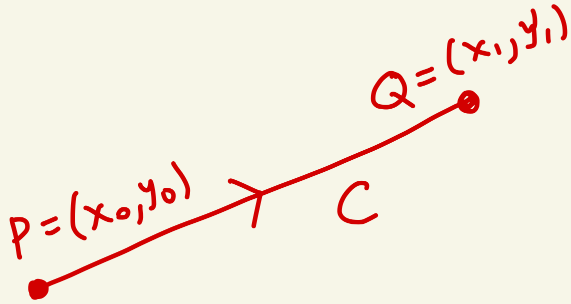


parameterizes a curve C where $a \le t \le b$.

$\boxed{\text{Calc II formula}}$

The line segment C between $P=(x_0,y_0)$ and $Q=(x_1,y_1)$ can be parameterized by

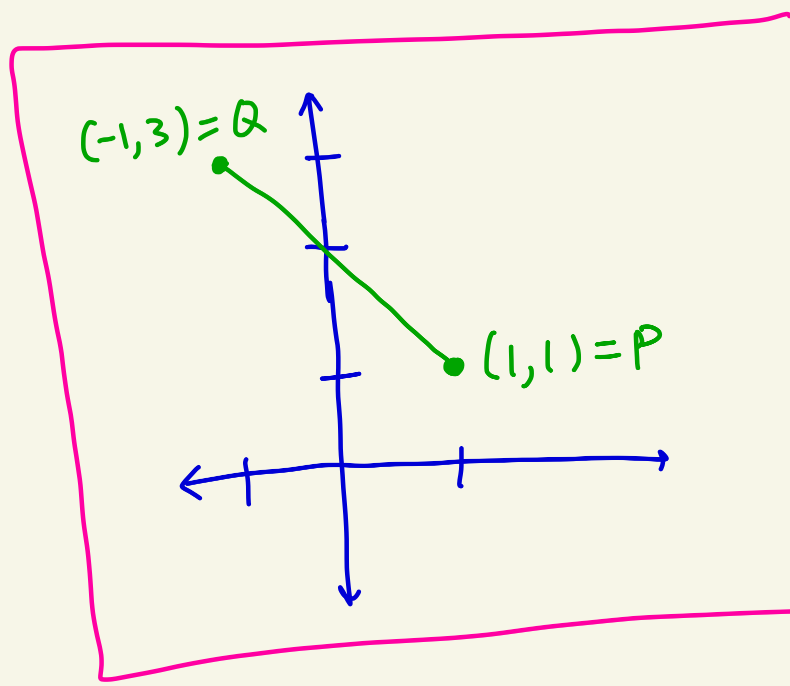
$$x = x_0 + (x_1 - x_0)t$$
$$y = y_0 + (y_1 - y_0)t$$
$$0 \le t \le 1$$



Corresponding to

$$\vec{r}(t) = \langle x_0 + (x_1 - x_0)t,\ y_0 + (y_1 - y_0)t \rangle$$

<u>Ex:</u> The line segment C between $(1,1)$ and $(-1,3)$ is given by



$$x = 1 + (-1-1)t = 1 - 2t$$
$$y = 1 + (3-1)t = 1 + 2t$$

$0 \le t \le 1$.

This corresponds to

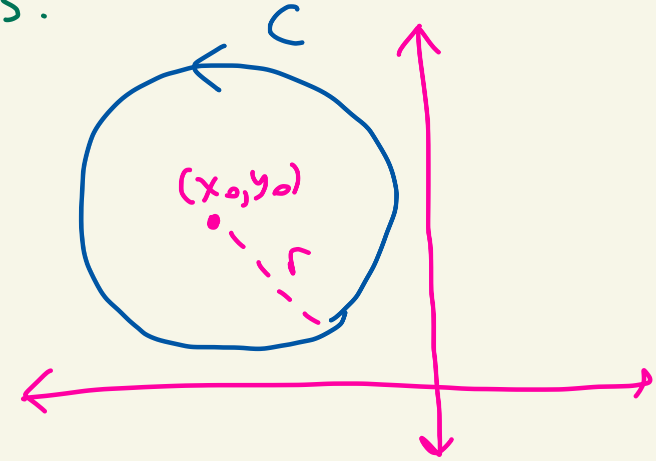
$$\vec{r}(t) = \langle 1 - 2t, 1 + 2t \rangle$$

for $0 \le t \le 1$.

$\boxed{\text{Calc II formula}}$

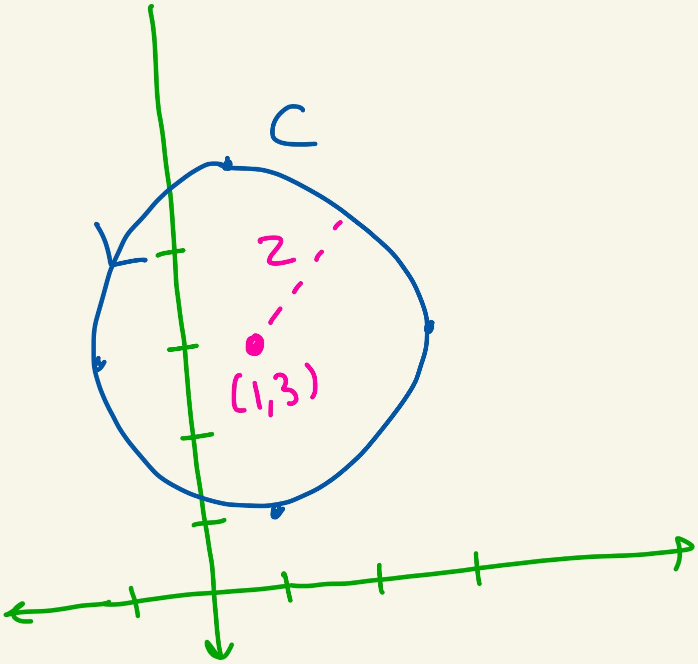
The circle, oriented counterclockwise, with radius r centered at (x_0, y_0) can be parameterized as:

$$x = x_0 + r\cos(t)$$
$$y = y_0 + r\sin(t)$$
$$0 \le t \le 2\pi$$



Ex: The circle C with radius 2 centered at $(1,3)$ oriented counterclockwise is parameterized by

$$x = 1 + 2\cos(t)$$
$$y = 3 + 2\sin(t)$$
$$0 \le t \le 2\pi$$



Calc II

Recall that a curve C given by $\vec{r}(t)$ on $a \le t \le b$ is called <u>smooth</u> if

① $\vec{r}\,'(t)$ exists for $a \le t \le b$ ← no abrupt edges

② $\vec{r}\,'(t) \ne \vec{0}$ for $a \le t \le b$ ← no stops on curve

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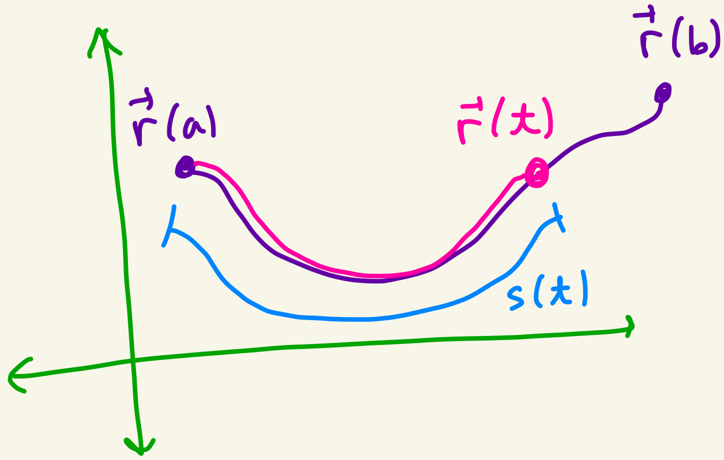
Calc II

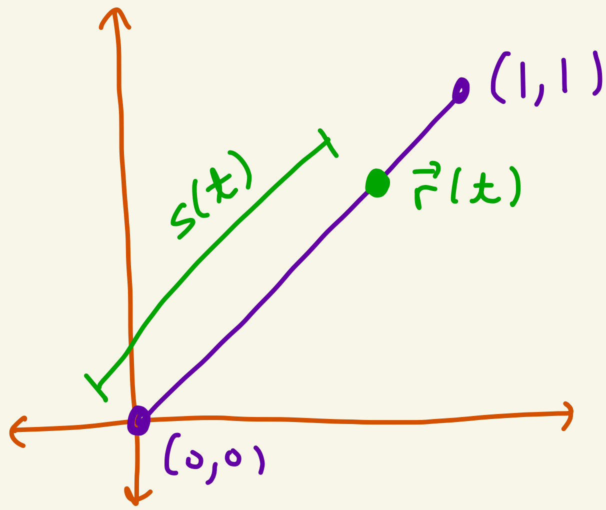
If C is a smooth curve described by $\vec{r}(t) = \langle x(t), y(t) \rangle$ for $a \le t \le b$, then the arclength along C from $\vec{r}(a)$ to $\vec{r}(t)$ is

$$\underbrace{s(t)}_{\text{s is used for arclength}} = \int_a^t |\vec{r}\,'(u)|\, du$$



$$= \int_a^t \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Ex: Let C be the line segment from $(0,0)$ to $(1,1)$.



We have the parameterization

$$\vec{r}(t) = \langle t, t \rangle$$

$x = 0 + (1-0)t$ $y = 0 + (1-0)t$

where $0 \le t \le 1$.

Then,

$$\vec{r}\,'(t) = \langle 1, 1 \rangle$$

The arc length from $(0,0)$ to $\vec{r}(t)$ is

$$s(t) = \int_0^t |\vec{r}\,'(u)|\, du$$

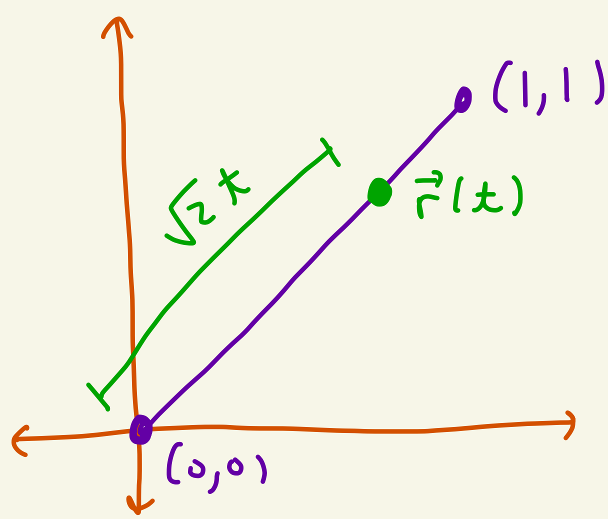
$$= \int_0^t \sqrt{1^2 + 1^2}\, du$$

$$\boxed{\begin{gathered}\vec{r}\,'(t) = \langle 1,1 \rangle \\ |\vec{r}\,'(t)| = \sqrt{1^2+1^2}\end{gathered}}$$

$$= \sqrt{2}\, u \Big|_0^t$$

$$= \sqrt{2}\,(t - 0)$$

$$= \sqrt{2}\, t$$



We can re-parameterize the curve C in terms of arc length as follows:

$$\left.\begin{aligned} &\vec{r}(t) = \langle t, t \rangle \\ &0 \le t \le 1 \\ &s = \sqrt{2}\, t \end{aligned}\right] \text{have so far}$$

Solve for t to get

$$t = \frac{s}{\sqrt{2}}$$

Plug into $\vec{r}$ to get

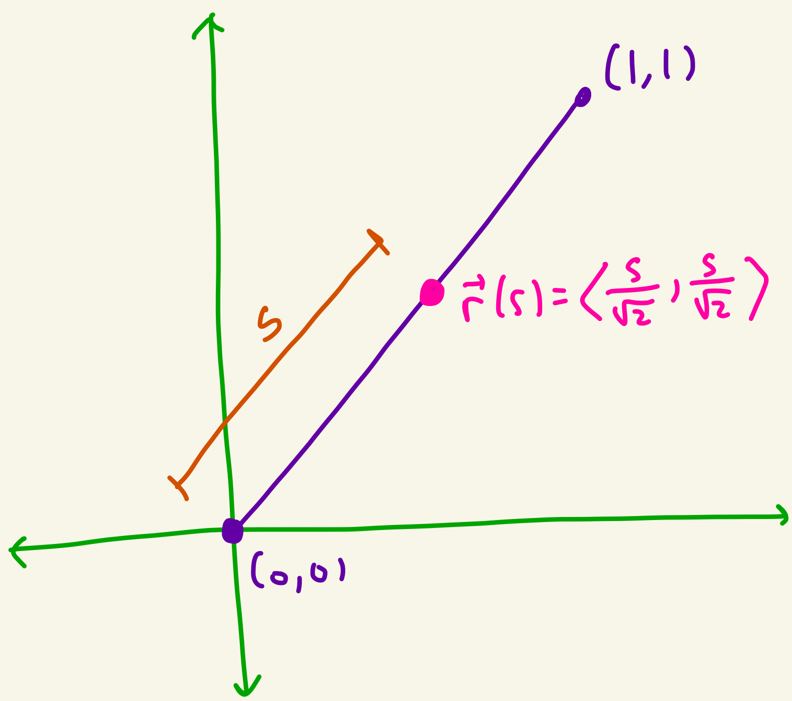
$$\vec{r}(s) = \left\langle \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle$$

where

$$0 \le s \le \sqrt{2}$$

Plug $t=0$ into $s=\sqrt{2}\,t$ to get $s=0$

Plug $t=1$ into $s=\sqrt{2}\,t$ to get $s=\sqrt{2}$



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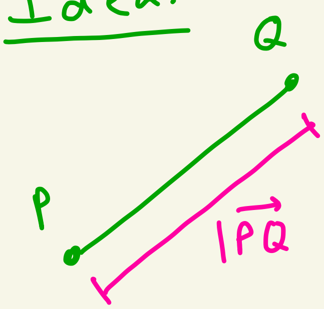
Note:

The line segment C between $P=(x_0,y_0)$ and $Q=(x_1,y_1)$ with length $|\overrightarrow{PQ}|$ can be parameterized in terms of arc length as follows:

$$r(s) = \left\langle x_0 + \frac{1}{|\overrightarrow{PQ}|}(x_1-x_0)s,\ y_0 = \frac{1}{|\overrightarrow{PQ}|}(y_1-y_0)s \right\rangle$$

where $0 \le s \le |\overrightarrow{PQ}|$.

Idea:



$$\vec{r}(s) = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}s$$

That is,

$$x = x_0 + \frac{1}{|\overrightarrow{PQ}|}(x_1-x_0)s$$

$$y = y_0 + \frac{1}{|\overrightarrow{PQ}|}(y_1-y_0)s$$

where $0 \le s \le |\overrightarrow{PQ}|$.

Ex: $P=(1,0)$, $Q=(1,2)$

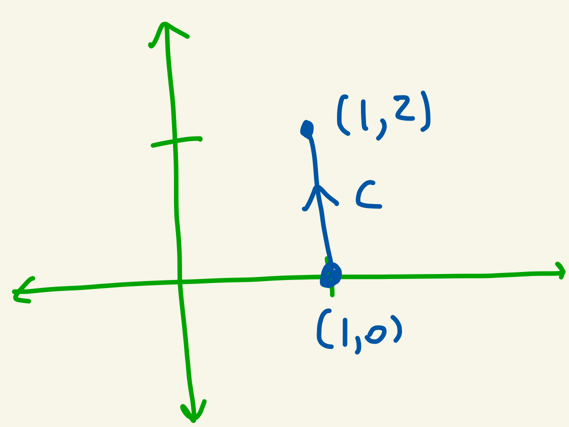
$|\overrightarrow{PQ}| = |\langle 1-1, 0-2\rangle| = |\langle 0,-2\rangle| = \sqrt{0^2+(-2)^2} = 2$

C is parameterized by

$x = 1 + \frac{1}{2}(1-1)s = 1$

$y = 0 + \frac{1}{2}(2-0)s = s$

$0 \le s \le 2$

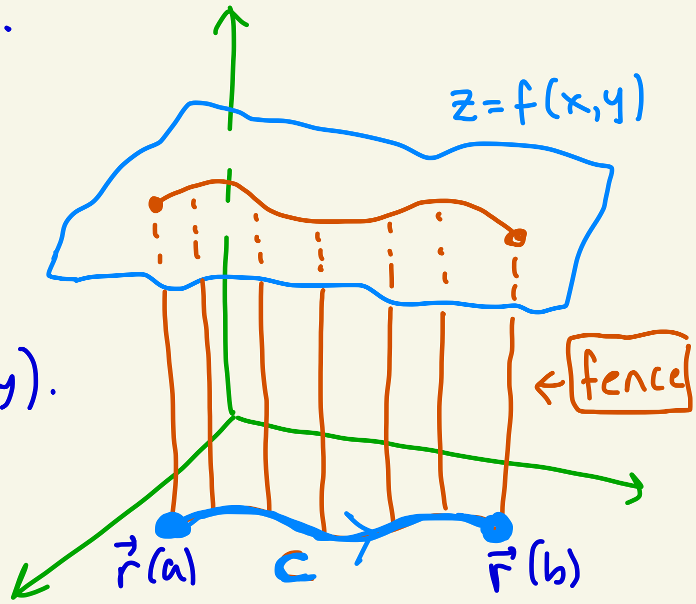


Part 2: Line integrals

Let C be a smooth curve parameterized in terms of arc length as $\vec{r}(s) = \langle x(s), y(s) \rangle$ for $a \le s \le b$.

We start with a 2d-curve. Later we do a 3d-curve

Consider a function $f(x,y)$.
When $f(x,y) \geq 0$ we want an integral that finds the area of the "fence" above C and under $z=f(x,y)$.



Subdivide C into n sub-arcs by subdividing the interval $[a,b]$ as

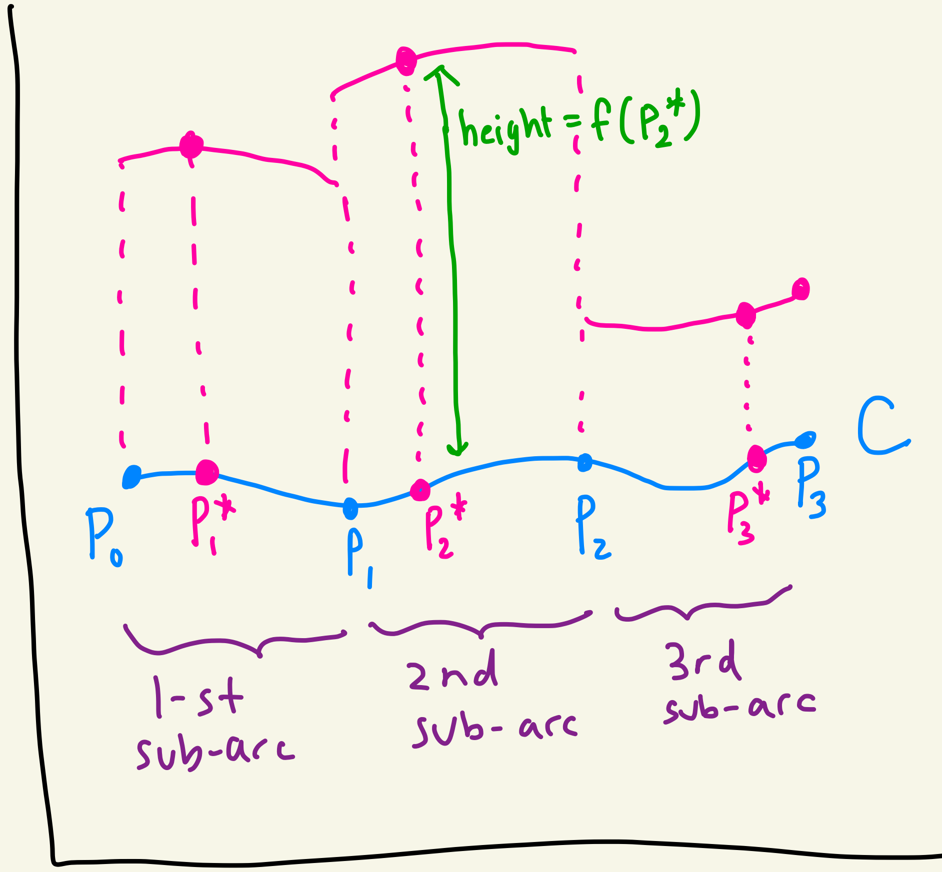
$$a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b$$

corresponding to n points $P_0, P_1, P_2, \ldots, P_n$ along C where $P_i = (x(s_i), y(s_i))$

Pick n points

$P_1^*, P_2^*, \ldots, P_n^*$

where P_i^* lies on the sub-arc between P_{i-1} and P_i.

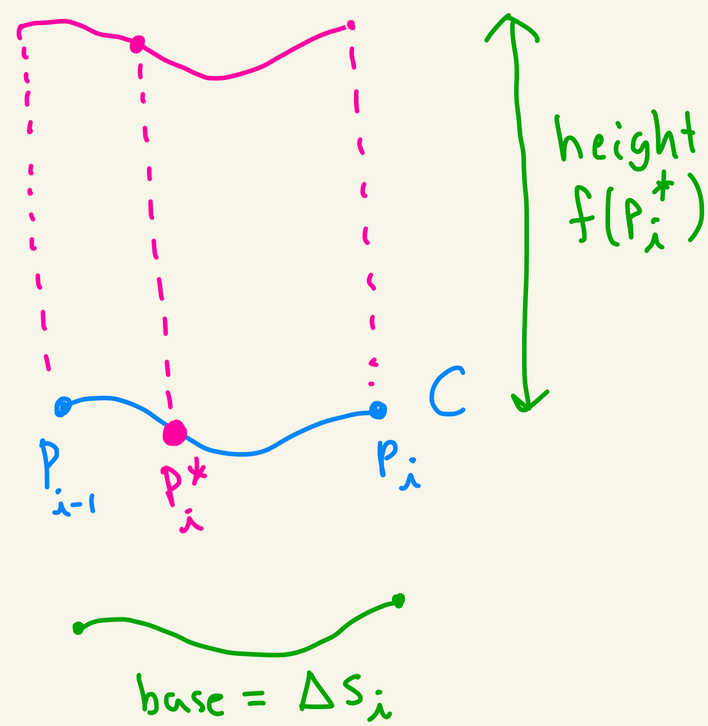


Let Δs_i represent the arclenth of the ith subarc.

On the i-th sub-arc the product

$$f(P_i^*) \cdot \Delta s_i$$

represents the "area" of the approximation to the area of the fence on the ith sub-arc



So the area of the "fence" is approximated by

$$\sum_{i=1}^{n} f(P_i^*) \Delta s_i$$

Let Δ be the maximum value of the lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$ of the subarcs.

The <u>line integral</u> of f over C is defined to be

$$\int_C f(x,y)\,ds = \lim_{\substack{n\to\infty \\ \Delta\to 0}} \sum_{i=1}^{n} f(P_i^*)\,\Delta s_i$$

if this limit exists.

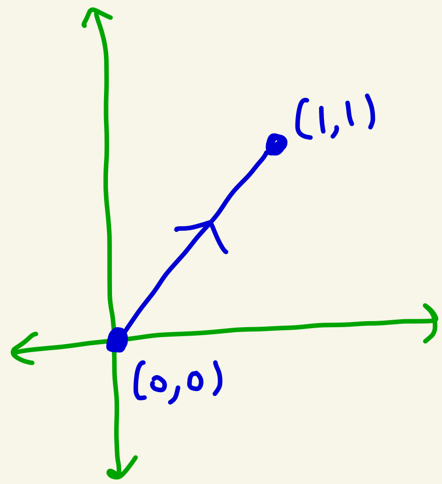
To evaluate the above integral we calculate:

$$\int_C f(x,y)\,ds = \underbrace{\int_C f(x(s),y(s))\,ds}$$

plug in the parameterization of C in terms of $x(s), y(s)$ and integrate with respect to s

Ex: Calculate

$$\int_C (x+y)\, ds$$



where C is the line segment from $(0,0)$ to $(1,1)$.

Previously we saw that C is described by

$$x(s) = \frac{s}{\sqrt{2}} \qquad y(s) = \frac{s}{\sqrt{2}}$$

where $0 \le s \le \sqrt{2}$.

So,

$$\int_C (x+y)\, ds = \int_0^{\sqrt{2}} \left(\frac{s}{\sqrt{2}} + \frac{s}{\sqrt{2}}\right) ds = \frac{2}{\sqrt{2}} \int s\, ds$$

$$= \frac{2}{\sqrt{2}}\left(\frac{s^2}{2}\right)\Bigg|_0^{\sqrt{2}} = \frac{2}{\sqrt{2}}\left(\frac{(\sqrt{2})^2}{2} - \frac{0^2}{2}\right) = \frac{2}{\sqrt{2}} = \boxed{\sqrt{2}}$$

We get some usual properties:

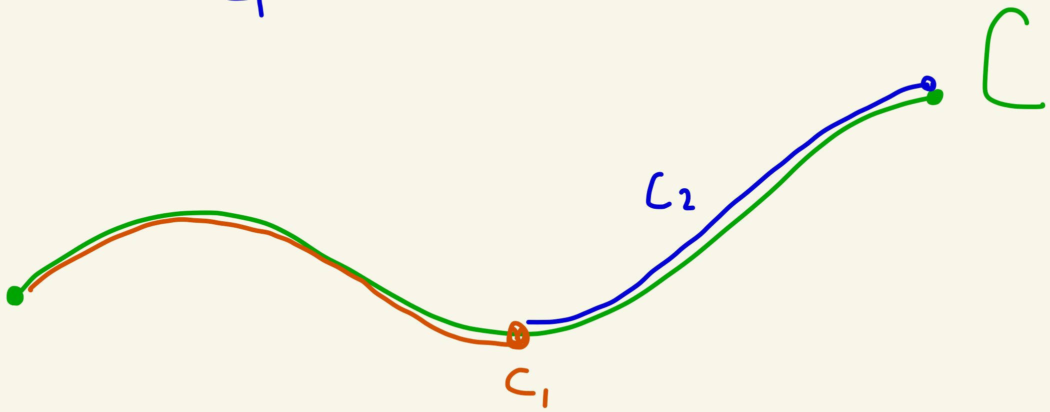
① $$\int_C [f(x,y)+g(x,y)]\,ds = \int_C f(x,y)\,ds + \int_C g(x,y)\,ds$$

② If α is a constant, then

$$\int_C \alpha f(x,y)\,ds = \alpha\left[\int_C f(x,y)\,ds\right]$$

③ If C is formed by first going along C_1 and then along C_2 then

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds$$



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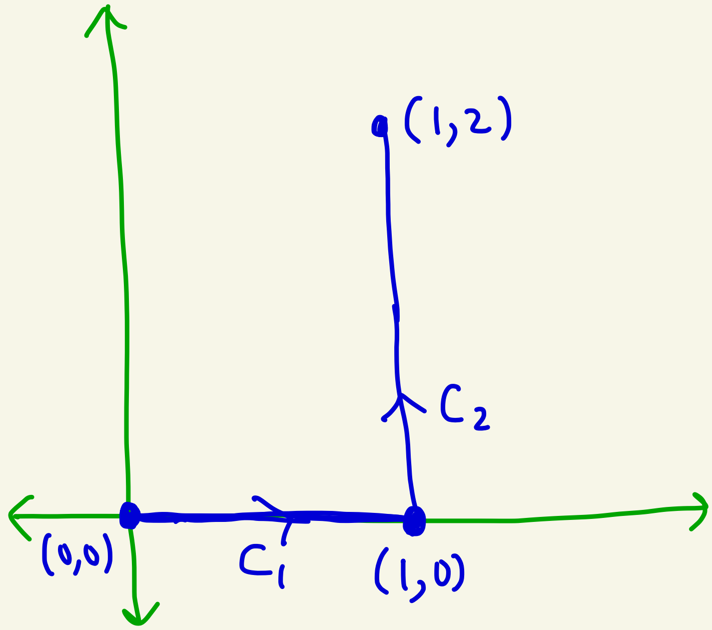
Ex: Calculate

$$\int_C (x+y)\,ds$$

where C is broken into C_1 and C_2 as shown.



We have:

$$\int_{C_1}(x+y)\,ds = \int_0^1 (s+0)\,ds = \frac{s^2}{2}\Big|_0^1 = \frac{1}{2}$$

$x = s$
$y = 0$
$0 \le s \le 1$

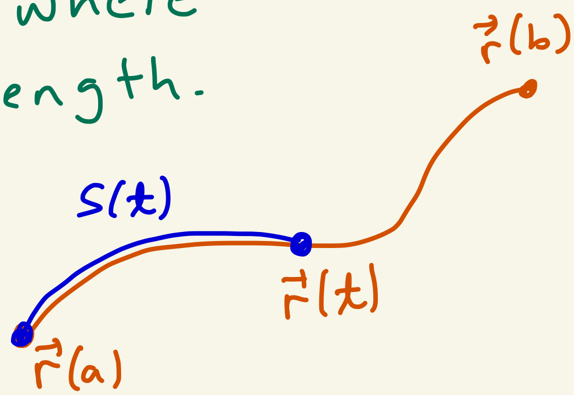
$$\int_{C_2}(x+y)\,ds = \int_0^2 (1+s)\,ds = s+\frac{s^2}{2}\Big|_0^2 = \left(2+\frac{2^2}{2}\right)-(0) = 4$$

$x = 1$
$y = s$
$0 \le s \le 2$

So,

$$\int_C (x+y)\,ds = \frac{1}{2}+4 = \boxed{4.5}$$

Suppose we want to parameterize C by $\vec{r}(t) = \langle x(t), y(t) \rangle$ where $a \leq t \leq b$ and t is not arc-length.



Then, (from Calc II),

the arclength on C from $\vec{r}(a)$ to $\vec{r}(t)$ is

$$s(t) = \int_a^t |\vec{r}\,'(u)|\,du$$

and $s'(t) = |r'(t)|$

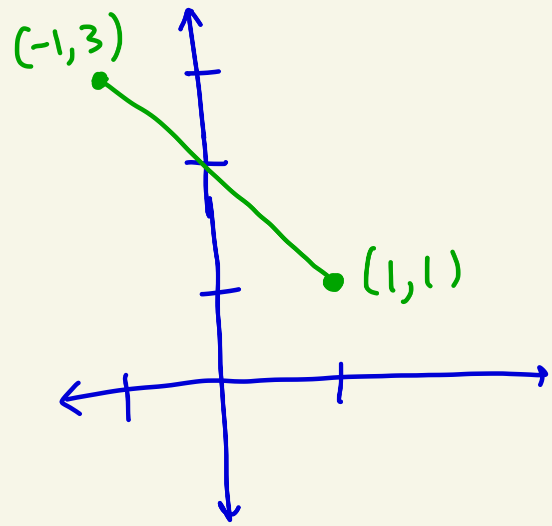
So,

$$\begin{aligned} ds &= s'(t)\,dt \\ &= |r'(t)|\,dt \\ &= |\langle x'(t), y'(t) \rangle|\,dt \\ &= \sqrt{(x'(t))^2 + (y'(t))^2}\,dt \end{aligned}$$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t)) \cdot |\vec{r}\,'(t)|\,dt$$

$$= \int_a^b f(x(t),y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

Ex: Let C be the line segment from $(1,1)$ to $(-1,3)$



C is given by

$$x = 1 + (-1-1)t = 1 - 2t$$
$$y = 1 + (3-1)t = 1 + 2t$$
$$0 \le t \le 1$$

Then,

$$x' = -2$$
$$y' = 2$$
$$ds = \sqrt{(x')^2 + (y')^2}\, dt = \sqrt{(-2)^2 + (2)^2}\, dt = \sqrt{8}\, dt$$

So,

$$\int_C (2x-y)\, ds = \int_0^1 \underbrace{\big(2(1-2t) - (1+2t)\big)}_{2x-y} \underbrace{\sqrt{8}\, dt}_{ds}$$

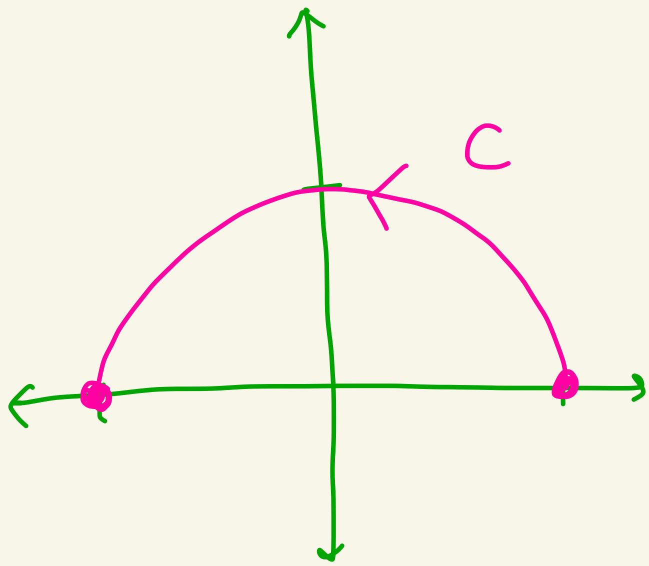
$$= \int_0^1 \sqrt{8}\,[1-2t]\, dt = \sqrt{8}\left[t - t^2 \Big|_0^1 \right]$$

$$= \sqrt{8}\left[(1-1^2) - (0-0^2) \right] = 0$$

Ex: Evaluate $\int_C (2+x^2y)\,ds$ where C is the upper half of the unit circle, oriented counter-clockwise.

We have:

$$x(t) = \cos(t)$$
$$y(t) = \sin(t)$$
$$0 \le t \le \pi$$



So:

$$\int_C (2+x^2y)\,ds = \int_0^{\pi} \underbrace{\left(2+\cos^2(t)\sin(t)\right)}_{2+x^2y} \underbrace{\sqrt{(-\sin(t))^2+(\cos(t))^2}\,dt}_{ds}$$

$$= \int_0^{\pi} \left(2+\cos^2(t)\sin(t)\right) dt$$

$$= \int_0^{\pi} 2\,dt + \int_0^{\pi} \cos^2(t)\sin(t)\,dt$$

$$= 2\pi - \int_1^{-1} u^2\,du = 2\pi - \left(\frac{u^3}{3}\Big|_1^{-1}\right)$$

$u = \cos(t)$
$du = -\sin(t)dt$
$-du = \sin(t)dt$
$t=0 \longrightarrow u=1$
$t=\pi \longrightarrow u=-1$

$$= 2\pi - \left(\frac{(-1)^3}{3} - \frac{1}{3}\right)$$

$$= 2\pi + \frac{2}{3}$$

Note: Suppose C is a smooth 3-dimensional curve given by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

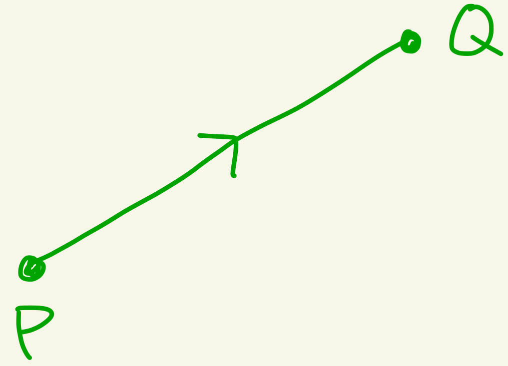
Then all the same ideas apply and

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t),z(t)) \cdot |\vec{r}\,'(t)|\,dt$$

$$= \int_a^b f(x(t),y(t),z(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

Recall: The line segment from $P = (x_0, y_0, z_0)$ to $Q = (x_1, y_1, z_1)$ is given by

$$x = x_0 + (x_1 - x_0)t$$
$$y = y_0 + (y_1 - y_0)t$$
$$z = z_0 + (z_1 - z_0)t$$
$$0 \le t \le 1$$



Ex: Evaluate $\int_C (xy+2z)\,ds$ where C is the line segment from $P=(1,0,0)$ to $Q=(0,1,1)$.

C is given by

$$x = 1+(0-1)t = 1-t$$
$$y = 0+(1-0)t = t$$
$$z = 0+(1-0)t = t$$
$$0 \le t \le 1$$

So,

$$\int_C (xy+2z)\,ds = \int_0^1 \left[(1-t)t+2t\right]\sqrt{(-1)^2+(1)^2+(1)^2}\,dt$$

$$\boxed{\begin{array}{l} x=1-t,\ x'=-1 \\ y=t,\ y'=1 \\ z=t,\ z'=1 \end{array}}$$

$$= \sqrt{3}\int_0^1 \left[-t^2+3t\right]dt = \sqrt{3}\left[-\frac{t^3}{3}+3\frac{t^2}{2}\right]\Bigg|_0^1$$

$$= \sqrt{3}\left[-\frac{1}{3}+\frac{3}{2}\right] = \boxed{\frac{7}{6}\sqrt{3}}$$